Introduction to the Fourier transform

Frames 1 to 53

Learning outcomes

When you have completed this Programme you will be able to:

- Convert a trigonometric Fourier series into a doubly infinite sum of complex exponentials
- Derive the complex Fourier series of a function that satisfies Dirichlet's conditions
- Recognise the function sinc (t)
- Separate a discrete complex spectrum into an amplitude spectrum and a phase spectrum
- State Fourier's integral theorem in terms of complex exponentials
- Define and derive the Fourier transform of a function satisfying Dirichlet's conditions
- Separate a continuous complex spectrum into an amplitude spectrum and a phase spectrum
- Recognise the functions $\Pi_a(t)$ and $\Lambda_a(t)$ and derive their Fourier transforms along with those of the Dirac delta and the Heaviside unit step
- Recognise alternative forms of the function–transform pair
- Reproduce a collection of properties of the Fourier transform
- Evaluate the convolution of two functions and describe its Fourier transform
- Derive the Fourier sine and cosine transformations.

Complex Fourier series

1

Introduction

In the previous Programme we saw how a periodic function can be represented by an infinite sum of periodic, trigonometric harmonics. Each harmonic has a definite frequency which is an integer multiple of the fundamental frequency. A non-periodic function can be similarly represented, not as a sum but as an integral over a continuous range of frequencies. Before we do this, however, we shall convert the infinite Fourier series in terms of sines and cosines into a doubly infinite series involving complex exponentials.

Complex exponentials

Recall the exponential form of a complex number and its relationship to the polar form, namely

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

From this equation we can see that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

and so

$$\cos(-\theta) + j\sin(-\theta) = e^{-j\theta} = \cos\theta - j\sin\theta$$

Using these two equations we can find the complex exponential form of the trigonometric functions as

$\cos\theta = \dots\dots\dots\dots$ and $\sin\theta = \dots\dots\dots\dots$

2

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \quad \text{and}\ \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}}$$

Because

$$\cos\theta + j\sin\theta = e^{j\theta}\text{and } \cos\theta - j\sin\theta = e^{-j\theta}$$

so adding these two equations gives

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \text{ that is } \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \qquad (1)$$

and subtracting the two equations gives

$$2j\sin\theta = e^{j\theta} - e^{-j\theta} \text{ that is } \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} \qquad (2)$$

These two equations permit us to develop an alternative representation of a Fourier series.

▶

In the previous Programme we found that the Fourier series of the piecewise continuous function $f(t)$ with piecewise continuous derivative and where $f(t+T)=f(t)$ is given as

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t) \qquad (3)$$

where $\omega_0=\dfrac{2\pi}{T}$ and where $a_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}f(t)\cos n\omega_0 t\,dt$

and $b_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}f(t)\sin n\omega_o t\,dt$

Now, if we substitute the right-hand sides of equations (1) and (2) into equation (3) we obtain

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\ldots\ldots\ldots\ldots\right\}e^{jn\omega_0 t}+\left\{\ldots\ldots\ldots\ldots\right\}e^{-jn\omega_0 t}\right)$$

3

$$\boxed{f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\frac{a_n-jb_n}{2}\right\}e^{jn\omega_0 t}+\left\{\frac{a_n+jb_n}{2}\right\}e^{-jn\omega_0 t}\right)}$$

Because

$$\begin{aligned}f(t)&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2}+b_n\frac{e^{jn\omega_0 t}-e^{-jn\omega_0 t}}{2j}\right)\\&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\frac{a_n+b_n/j}{2}\right\}e^{jn\omega_0 t}+\left\{\frac{a_n-b_n/j}{2}\right\}e^{-jn\omega_0 t}\right)\\&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\frac{a_n-jb_n}{2}\right\}e^{jn\omega_0 t}+\left\{\frac{a_n+jb_n}{2}\right\}e^{-jn\omega_0 t}\right)\end{aligned}$$

In the next frame we shall make some notational changes to simplify this expression

4

If we now define $c_n = \dfrac{a_n - jb_n}{2}$ so that the complex conjugate of c_n is $c_n^* = \dfrac{a_n + jb_n}{2}$ we can write this sum as

$$f(t) = c_0 + \sum_{n=1}^{\infty}(c_n e^{jn\omega_0 t} + c_n^* e^{-jn\omega_0 t})$$

Note that we have taken $b_0 = 0$. There is no problem about this. There is no term $\sin 0\omega_0 t$ in the Fourier series and so $b_0 = 0$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_n^* e^{-jn\omega_0 t}$$

For notational convenience we denote c_n^* by c_{-n}. This means that $a_{-n} = a_n$ and $b_{-n} = -b_n$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

As n ranges from 1 to ∞ so $-n$ ranges from -1 to $-\infty$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t}$$

Notice the reversed order of summation in the first sum

$$= \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t}$$

Combining all three terms into the *doubly infinite sum*

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $c_n = \dfrac{a_n - jb_n}{2} = \dfrac{2}{2T}\displaystyle\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)\,dt$. That is $c_n = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$.

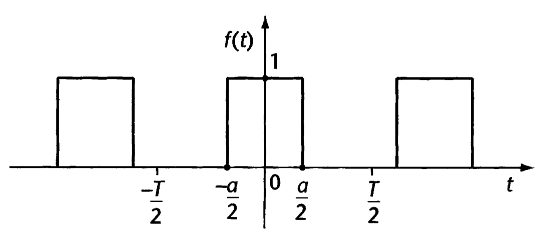
In the next frame we shall look at some examples

5

Example 1

To find the complex Fourier series for the function

$$f(t) = \begin{cases} 0 & -T/2 < t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t < T/2 \end{cases} \quad \text{where } f(t+T) = f(t)$$



we proceed as on the next page.

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} && \text{where } \omega_0 = \frac{2\pi}{T} \text{ and} \\
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} \\
&= \frac{1}{T}\int_{-a/2}^{a/2} e^{-jn\omega_0 t}\,\mathrm{d}t && \text{Because } f(t) = 1 \text{ for } -a/2 < t < a/2 \\
&= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-a/2}^{a/2} && \text{Provided } n \neq 0 \\
&= \left(\frac{e^{-jn\omega_0 a/2} - e^{jn\omega_0 a/2}}{-j2n\pi}\right) && \text{Since } \omega_0 = \frac{2\pi}{T} \\
&= \frac{\sin n\omega_0 a/2}{n\pi} && \text{Recall that } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \\
&= \frac{\sin n\pi a/T}{n\pi} && \text{Since } \omega_0 = \frac{2\pi}{T} \\
&= \frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) && \text{Provided } n \neq 0
\end{aligned}
$$

When $n = 0$

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \frac{1}{T}\int_{-a/2}^{a/2} \mathrm{d}t = \frac{a}{T}$$

Therefore

$$f(t) = \frac{a}{T} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)e^{jn\omega_0 t}$$

In the next frame we shall look at the same function retarded by half the width of the peak

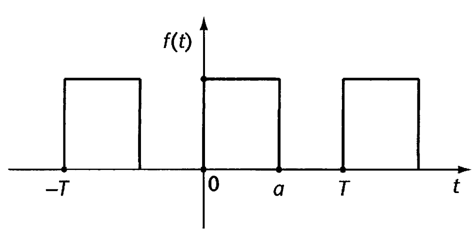
---

6

Example 2

To find the complex Fourier series for the function

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & a < t < T \end{cases} \quad \text{where } f(t+T) = f(t)$$



We find that, for $n \neq 0$,

$c_n = \ldots\ldots\ldots\ldots$

7

$$\boxed{c_n = e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)}$$

Because

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad \text{where } \omega = \frac{2\pi}{T} \text{ and}$$

$$\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt \\
&= \frac{1}{T}\int_0^a e^{-jn\omega_0 t}\,dt \\
&= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^a \qquad \text{Provided } n \neq 0 \\
&= \left(\frac{e^{-jn\omega_0 a} - 1}{-j2n\pi}\right) \\
&= e^{-jn\omega_0 a/2}\left(\frac{e^{-jn\omega_0 a/2} - e^{jn\omega_0 a/2}}{-j2n\pi}\right) \\
&= e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) \qquad \text{Provided } n \neq 0
\end{aligned}$$

To finish

$c_0 = \dots\dots\dots\dots$

8

$$\boxed{c_0 = \frac{a}{T}}$$

Because

$$\begin{aligned}
c_0 &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \\
&= \frac{1}{T}\int_0^a dt = \frac{a}{T}
\end{aligned}$$

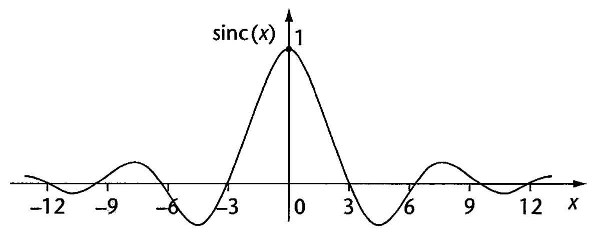
Therefore

$$f(t) = \frac{a}{T} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)e^{jn\omega_0 t}$$

Next frame

9

Before we move on, consider the expression $\dfrac{\sin n\pi a/T}{n\pi a/T}$ that occurs in both of these examples. This is an example of a commonly occurring expression $\dfrac{\sin x}{x}$ which has the special name $\operatorname{sinc}(x)$. Notice that $\operatorname{sinc}(0)$ is not defined. However, because $\underset{x\to 0}{Lim}\operatorname{sinc}(x) = \underset{x\to 0}{Lim}\dfrac{\sin x}{x} = 1$ we define $\operatorname{sinc}(0) = 1$.



This means that c_0 can be incorporated into the summations so the solutions to Examples 1 and 2 become

$$f(t) = \sum_{n=-\infty}^{\infty} (a/T)\operatorname{sinc}(n\pi a/T)e^{jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} (a/T)e^{-jn\pi a/T}\operatorname{sinc}(n\pi a/T)e^{jn\omega_0 t} \quad \text{respectively.}$$

Now let's compare these two results

Complex spectra

10

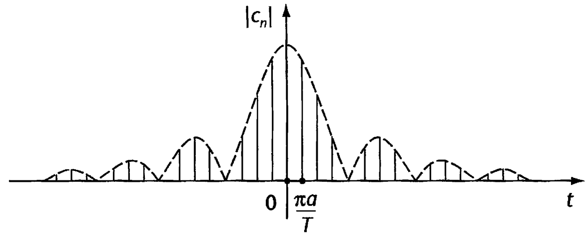
The coefficients c_n in the first example are real numbers whereas in the second example they are complex numbers. In general, the c_n are complex numbers and can be written as

$c_n = |c_n|e^{j\phi_n}$ where, in the last example $|c_n| = \dfrac{a}{T}\left|\dfrac{\sin n\pi aT}{n\pi T}\right|$

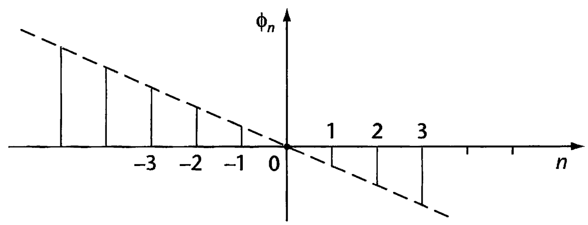
and $\phi_n = -n\pi a/T$.

These complex coefficients constitute a **discrete complex spectrum** where c_n represents the *spectral coefficient* of the nth harmonic. Each spectral coefficient couples an **amplitude spectrum** value $|c_n|$ and a **phase spectrum** value ϕ_n. The amplitude spectrum tells us the magnitude of each of the harmonic components and has, for both examples, the graph shown on the next page.

▶



The phase spectrum $\phi_n = -n\pi a/T$ tells us the phase of each harmonic relative to the fundamental harmonic frequency ω_0.



The phase spectrum of the first example is zero for all n and tells us that each harmonic is in phase with the fundamental harmonic. The phase spectrum of the second example, which is a retarded form of the first example, tells us that the nth harmonic is shifted out of phase from the fundamental harmonic by $n\omega_0$.

Next frame

The two domains

11

A periodic waveform and its spectrum are described in different terms. The waveform is described in terms of behaviour in time whereas the spectrum is described in terms of behaviour relative to frequency. Thus time and frequency form two domains of definition of our functions and whatever information can be gleaned from within one domain can equally be gleaned from within the other. For example, the *power content* of a periodic function $f(t)$ of period T is defined in the time domain as the mean square value of $f(t)$

$$\frac{1}{T}\int_{-T/2}^{T/2} (f(t))^2 \, dt$$

Within the frequency domain the power content is given as

. . . .

12

$$\boxed{\sum_{n=-\infty}^{\infty} |c_n|^2}$$

Because

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2}(f(t))^2\,\mathrm{d}t &= \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right)f(t)\,\mathrm{d}t\\ &= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}\,\mathrm{d}t\\ &= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j(-n)\omega_0 t}\,\mathrm{d}t\\ &= \sum_{n=-\infty}^{\infty} c_n c_{-n} = \sum_{n=-\infty}^{\infty} c_n c_n^*\\ &= \sum_{n=-\infty}^{\infty} |c_n|^2\end{aligned}$$

So the power content can be obtained from either domain.

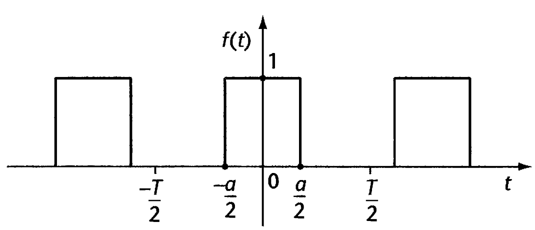
Next frame

Continuous spectra

13

Of interest in the analysis of periodic functions is the behaviour of the Fourier series as the period increases without limit. Consider Example 1 from Frame 5

$$f(t) = \begin{cases} 0 & -T/2 < t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t < T/2 \end{cases} \quad \text{where } f(t+T) = f(t)$$



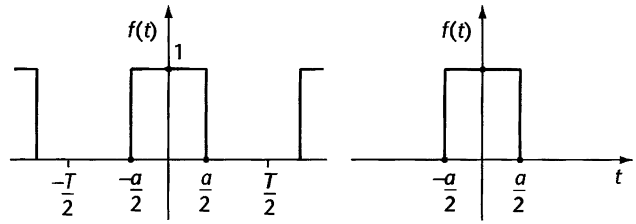
which has the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T} \text{ and where } c_n = \left(\frac{a}{T}\right)\frac{\sin\left(\frac{n\pi a}{T}\right)}{\frac{n\pi a}{T}}$$

As the period increases the separation between the pulses increases and in the limit as $T \to \infty$ only .... remains and the resulting function is no longer ....

14

only a single pulse remains and the resulting function is no longer periodic



In the Fourier series the distance between neighbouring harmonics in the complex spectra is the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ and, in the limit as $T \to \infty$, so $\omega_0 \to 0$. This means that as the period increases the space between lines in the spectrum decreases so the spectrum lines come closer together and in the limit merge into a continuous spectrum. That is, for large T

$n\omega_0 = n\delta\omega$ and as $T \to \infty$ so $n\delta\omega \to \omega$

where ω is the continuous frequency variable. To see the effect of this on the general form of the Fourier series we start with

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T}$$

$$\text{and where } c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

Substituting the integral form of c_n into the sum gives

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(u)e^{-jn\omega_0 u}\,du\right]e^{jn\omega_0 t}$$

where u is a dummy variable in place of the variable t.

Now, $\omega_0 = \frac{2\pi}{T}$ and so

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-T/2}^{T/2} f(u)e^{-jn\omega_0 u}\,du\right]\omega_0 e^{jn\omega_0 t}$$

If T is large then $\omega_0 = \delta\omega$ and

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-T/2}^{T/2} f(u)e^{-jn\delta\omega u}\,du\right]e^{jn\delta\omega t}\delta\omega$$

▶

In the limit as $T \to \infty$ so $n\delta\omega \to \omega$, the sum becomes an integral and $\delta\omega$ becomes the differential $\mathrm{d}\omega$ giving

$$\begin{aligned} f(t) &= \int_{\omega=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{u=-\infty}^{\infty} f(u)e^{-j\omega u}\,\mathrm{d}u \right] e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{\infty} f(u)e^{-j\omega u}\,\mathrm{d}u \right] e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{\infty} f(u)e^{-j\omega u}\,\mathrm{d}u \end{aligned}$$

These two integrals form the conclusion of *Fourier's integral theorem.*

Next frame

Fourier's integral theorem

15

Given function $f(t)$ with derivative $f'(t)$ where

(a) $f(t)$ and $f'(t)$ are piecewise continuous in every finite interval

(b) $f(t)$ is absolutely integrable in $(\infty, -\infty)$, that is $\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t$ is finite

then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \text{ where } F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

The discrete harmonic values $n\omega_0$ of the periodic function are now replaced by the continuous harmonic variable ω and the discrete spectra $c_n = |c_n|e^{j\phi_n}$ are replaced by the *continuous spectra* $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$. $F(\omega)$ is referred to as the **Fourier transform** of $f(t)$ and can also be written as $\mathscr{F}(f(t))$. Deriving the Fourier transform of a function is then a matter of applying the second of these two integrals. The expressions $f(t)$ and $F(\omega)$ form a Fourier transform pair where $f(t)$ can be referred to as the inverse Fourier transform of $F(\omega)$. That is, $f(t) = \mathscr{F}^{-1}[F(\omega)]$.

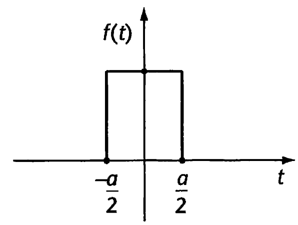
Next frame

16

Example 3

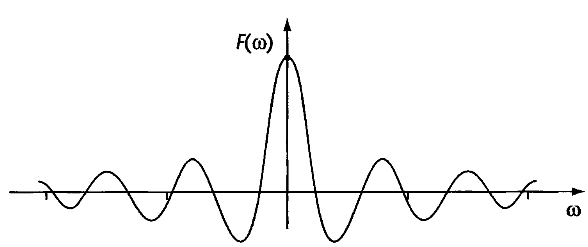
Find the Fourier transform of

$$f(t) = \begin{cases} 0 & t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t \end{cases}$$



$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-j\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-a/2}^{a/2} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-j\omega} \right) \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-2j\omega} \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin \omega a/2}{\omega} \\ &= \frac{a}{\sqrt{2\pi}} \frac{\sin \omega a/2}{\omega a/2} \\ &= \frac{a}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2) \end{aligned}$$

A plot of $F(\omega)$ produces the *continuous amplitude spectrum* of $f(t)$



Notice the similarity between the plots of $F(\omega)$ and the discrete spectrum of Frame 10. The lines in the discrete spectrum have merged to form a continuous spectrum while retaining the envelope of the discrete spectrum.

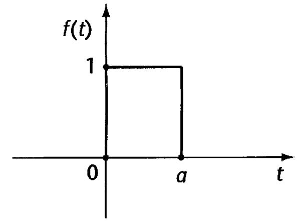
Now you try one

17

Example 4

The function of the previous example time delayed by $t = a/2$ units is

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & \text{otherwise} \end{cases}$$



And has the Fourier transform

$$F(\omega) = \ldots\ldots\ldots\ldots$$

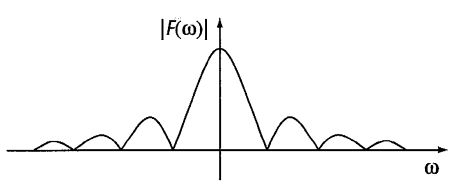
18

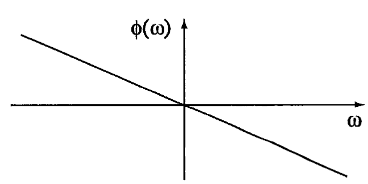
$$\boxed{F(\omega) = \frac{ae^{-j\omega a/2}}{\sqrt{2\pi}}\,\text{sinc}\,(\omega a/2)}$$

Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_0^a e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^a \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-j\omega a} - 1}{-j\omega}\right) \\
&= \frac{2}{\sqrt{2\pi}}e^{-j\omega a/2}\left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-2j\omega}\right) \\
&= \frac{2}{\sqrt{2\pi}}e^{-j\omega a/2}\left(\frac{\sin \omega a/2}{\omega}\right) \\
&= \frac{a}{\sqrt{2\pi}}e^{-j\omega a/2}\left(\frac{\sin \omega a/2}{\omega a/2}\right) \\
&= \frac{ae^{-j\omega a/2}}{\sqrt{2\pi}}\,\text{sinc}\,(\omega a/2)
\end{aligned}$$

Here $F(\omega)$ is a complex function so we write $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ where $|F(\omega)| = \left(a/\sqrt{2\pi}\right)\text{sinc}\,(\omega a/2)|$ is the *continuous amplitude spectrum* and $\phi(\omega) = -a\omega/2$ is the *continuous phase spectrum.*





Again, notice the similarity between the plots of $\phi(\omega)$ and the discrete phase spectrum of Frame 10. The lines in the discrete spectrum have merged to form a continuous spectrum while retaining the envelope of the discrete spectrum.

Next frame

Some special functions and their transforms

19

Even functions

If $f(t)$ is an even function then

$$f(-t) = f(t) \text{ and } f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

where

$$F(\omega) = \ldots\ldots\ldots\ldots \int_0^{\infty} f(t) \ldots\ldots\ldots\ldots \, dt$$

20

$$\boxed{F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt}$$

Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(t) e^{-j\omega t} \, dt + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) e^{-j\omega t} \, dt \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^{-\infty} f(t) e^{-j\omega t} \, dt + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) e^{-j\omega t} \, dt \\
&\quad \text{reversing the limits on the first integral} \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(-t) e^{j\omega t} \, d(-t) + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) e^{-j\omega t} \, dt \\
&\quad \text{changing the variable of integration in the first integral} \\
&\quad \text{from } t \text{ to } -t \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) \left[e^{j\omega t} + e^{-j\omega t} \right] d(t) \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt
\end{aligned}$$

Notice that if $f(t)$ is even then $F(\omega)$ is real.

Odd functions

If $f(t)$ is an odd function then

$$f(-t) = -f(t) \text{ and } f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

where

$$F(\omega) = \ldots\ldots\ldots\ldots \int_0^{\infty} f(t) \ldots\ldots\ldots\ldots \, dt$$

$$\boxed{F(\omega) = -j\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\,\mathrm{d}t}$$

Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(t)e^{-j\omega t}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t \\
&= -\frac{1}{\sqrt{2\pi}}\int_{0}^{-\infty} f(t)e^{-j\omega t}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t
\end{aligned}$$

reversing the limits on the first integral

$$= -\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(-t)e^{j\omega t}\,\mathrm{d}(-t) + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

changing the variable of integration in the first integral from t to $-t$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)\left[-e^{j\omega t} + e^{-j\omega t}\right]\mathrm{d}t \\
&= \frac{-2j}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)\sin\omega t\,\mathrm{d}t \\
&= -j\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t\,\mathrm{d}t
\end{aligned}$$

Notice that if $f(t)$ is odd then $F(\omega)$ is imaginary. An example will show the converse of these two results.

Example

Given that $\mathscr{F}f(t) = F(\omega) = A(\omega) + jB(\omega)$ where $A(\omega)$ and $B(\omega)$ are real functions of ω, then if

(a) $A(\omega) \neq 0$ and $B(\omega) = 0$ then $f(t)$ is an .... function
(b) $A(\omega) = 0$ and $B(\omega) \neq 0$ then $f(t)$ is an .... function

22

(a) $A(\omega) \neq 0$ and $B(\omega) = 0$ then $f(t)$ is an even function
(b) $A(\omega) = 0$ and $B(\omega) \neq 0$ then $f(t)$ is an odd function

Because

The Fourier transform is given as

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)[\cos\omega t - j\sin\omega t]\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt - j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt \\ &= A(\omega) + jB(\omega) \end{aligned}$$

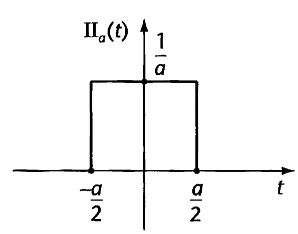
(a) If $\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt = 0$ then $f(t)\sin\omega t$ is odd. But $\sin\omega t$ is odd, so $f(t)$ must be even.

(b) If $\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = 0$ then $f(t)\cos\omega t$ is odd. But $\cos\omega t$ is even, so $f(t)$ must be odd.

Top-hat function

This function is a special form of the function met in Example 3 in Frame 16, and is defined by

$$f(t) = \begin{cases} 0 & t < -a/2 \\ 1/a & -a/2 < t < a/2 \\ 0 & a/2 < t \end{cases}$$



It is, because of its shape, referred to as the *top-hat* function and is denoted by the symbol $\Pi_a(t)$. It is a special form of the function in Example 3 because it has a unit area – width × height $= a \times (1/a) = 1$, or

$$\int_{-\infty}^{\infty} \Pi_a(t)\,dt = \int_{-a/2}^{a/2} (1/a)\,dt = \left[\frac{t}{a}\right]_{-a/2}^{a/2} = 1$$

The Fourier transform of the top-hat function is

$F(\omega) = \ldots\ldots\ldots\ldots$

$$\boxed{F(\omega) = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)}$$

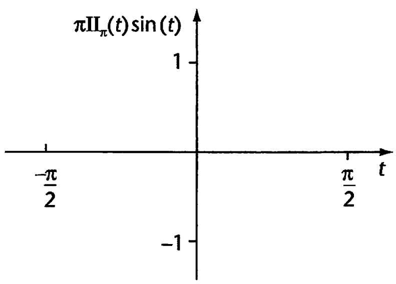
Because

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Pi_a(t) e^{-j\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} (1/a) e^{-j\omega t}\, dt \\ &= \frac{1}{a\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-j\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2) \end{aligned}$$

This function is useful in that it can be used to select any segment of any function. For example

$\pi\Pi_\pi(t)\sin t$

selects the segment of $\sin t$ between $\pm\pi/2$ and reduces the rest to zero.



So $\pi\Pi_\pi(t-\pi)\cos t$ selects the segment of $\cos t$ between .... and ....

24

$$\boxed{\pi/2 \text{ and } 3\pi/2}$$

Because

$$\Pi_\pi(t-\pi) = \begin{cases} 0 & t-\pi < -\pi/2 \\ 1/\pi & -\pi/2 < t-\pi < \pi/2 \\ 0 & \pi/2 < t-\pi \end{cases}$$

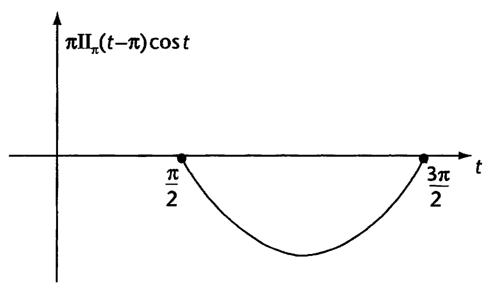
that is

$$\Pi_\pi(t-\pi) = \begin{cases} 0 & t < \pi/2 \\ 1/\pi & \pi/2 < t < 3\pi/2 \\ 0 & \pi/2 < t \end{cases}$$

and so

$$\pi\Pi_\pi(t-\pi)\cos t = \begin{cases} \cos t & \pi/2 < t < 3\pi/2 \\ 0 & \text{otherwise} \end{cases}$$

selects the segment of $\cos t$ between $\pi/2$ and $3\pi/2$.



The Dirac delta (refer to Programme 4, Frames 29*ff*)

In science and technology we often require to use the notion of a force that acts for a very brief interval of time. To simulate this mathematically we can use the unit-area pulse – the top-hat function. If we take the duration of this pulse to decrease while at the same time retaining a unit-area then in the limit we are led to the notion of the Dirac delta. That is

$$\underset{a \to 0}{Lim} \int_{-\infty}^{\infty} \{\Pi_a(t)\}\, dt = \underset{a \to 0}{Lim}\, 1 = 1$$

▶

Here as $a \to 0$ the width of the top-hat decreases as the height increases but all the while retaining the area beneath the top-hat as unity. It is this limit that we can use to justify the integral definition of the Dirac delta because

$$\underset{a\to 0}{Lim}\int_{-\infty}^{\infty}\{\Pi_a(t)\}\,\mathrm{d}t = \int_{-\infty}^{\infty}\underset{a\to 0}{Lim}\{\Pi_a(t)\}\,\mathrm{d}t = \int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = 1$$

and it is also in this sense that we accept the validity of the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,\mathrm{d}t = f(t_0)$$

because, like the top-hat function, it selects only that part of $f(t)$ over which it is non-zero, namely at $t = t_0$.

So if $f(t) = \delta(t)$ then $F(\omega) = \ldots\ldots\ldots\ldots$

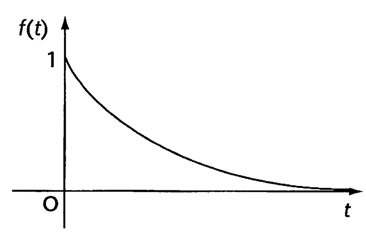
25

$$\boxed{\frac{1}{\sqrt{2\pi}}}$$

Because

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{e^{-j\omega 0}}{\sqrt{2\pi}} \qquad \text{because } \delta(t) = \delta(t-0) \\ &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

Try another.



The truncated exponential function

$$f(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases}$$

where $a > 0$ can be also expressed in the form $f(t) = e^{-at}u(t)$ and has the Fourier transform

$F(\omega) = \ldots\ldots\ldots\ldots$

26

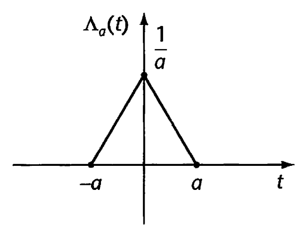
$$\boxed{F(\omega)=\frac{1}{\sqrt{2\pi}(a+j\omega)}}$$

Because

$$\begin{aligned}F(\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-at}u(t)e^{-j\omega t}\,dt\\&=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-(a+j\omega)t}\,dt\\&=\frac{1}{\sqrt{2\pi}(a+j\omega)}\end{aligned}$$

The triangle function

$$\Lambda_a(t)=\begin{cases}(a+t)/a^2 & -a<t<0\\(a-t)/a^2 & 0<t<a\\0 & |t|>a\end{cases}$$ Notice that this also has unit area



The Fourier transform of $\Lambda_1(t)$ is $F(\omega)=$ ....

27

$$\boxed{F(\omega)=\frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega/2)}$$

Because

$$\begin{aligned}F(\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Lambda_1(t)e^{-j\omega t}\,dt\\&=\frac{1}{\sqrt{2\pi}}\int_{-1}^{0}(1+t)e^{-j\omega t}\,dt+\frac{1}{\sqrt{2\pi}}\int_{0}^{1}(1-t)e^{-j\omega t}\,dt\\&=-\frac{1}{\sqrt{2\pi}}\int_{1}^{0}(1-t)e^{j\omega t}\,dt+\frac{1}{\sqrt{2\pi}}\int_{0}^{1}(1-t)e^{-j\omega t}\,dt\end{aligned}$$

changing the variable of integration in the first integral from t to $-t$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{1}(1-t)\cos\omega t\,dt$$ and integration by parts yields

$$\begin{aligned}&=\frac{2}{\sqrt{2\pi}}\left(\frac{1}{2}\frac{\sin^2(\omega/2)}{(\omega/2)^2}\right)\\&=\frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega/2)\end{aligned}$$

Alternative forms

28

It should be noted that there are a number of alternative forms for the Fourier transform – each dealing with a different location for the constant 2π. Other forms are

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \text{ where } F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

or

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

or, by absorbing the 2π in the exponential by defining $\omega = 2\pi\nu$

$$f(t) = \int_{-\infty}^{\infty} F(\nu)e^{j2\pi\nu t}\,\mathrm{d}\nu \text{ where } F(\nu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\nu t}\,\mathrm{d}t$$

We shall remain with our original form because it has the simplest exponential factor and we do not need to remember which integral has the constant in front of it and which does not.

Next frame

Properties of the Fourier transform

29

We now list a number of properties of the Fourier transform that are useful in their manipulation.

Linearity

If the Fourier transforms $\mathscr{F}(f_1(t)) = F_1(\omega)$ and $\mathscr{F}(f_2(t)) = F_2(\omega)$ then

$$\mathscr{F}(\alpha_1 f_1(t) + \alpha_2 f_2(t)) = \alpha_1\mathscr{F}(f_1(t)) + \alpha_2\mathscr{F}(f_2(t)) = \alpha_1 F_1(\omega) + \alpha_2 F_2(\omega)$$

where α_1 and α_2 are constants.

Example

The Fourier transform of $f(t) = 2\Pi_2(t) - 6\Lambda_2(t)$ is

$$F(\omega) = \ldots\ldots\ldots\ldots$$

30

$$\boxed{\sqrt{\frac{2}{\pi}}\operatorname{sinc}(\omega)(1-3\operatorname{sinc}(\omega))}$$

Because

If $f(t) = \Pi_2(t)$ then $F(\omega) = \dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}(\omega)$ and if $f(t) = \Lambda_2(t)$ then $F(\omega) = \dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega)$. Since $f(t) = 2\Pi_2(t) - 6\Lambda_2(t)$ then

$$\begin{aligned} F(\omega) &= \frac{2}{\sqrt{2\pi}}\operatorname{sinc}(\omega) - \frac{6}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega) \\ &= \sqrt{\frac{2}{\pi}}\operatorname{sinc}(\omega)(1-3\operatorname{sinc}(\omega)) \end{aligned}$$

Time shifting

If $\mathscr{F}(f(t)) = F(\omega)$ then $\mathscr{F}(f(t-t_0)) = e^{j\omega t_0}F(\omega)$

Example

The Fourier transform of $\Pi_2(t)$ is $\dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}(\omega)$ so, by the time shifting property, the Fourier transform of

$\Pi_2(t-5)$ is .... and of $\Pi_2(t+3)$ is ....

31

$$\boxed{\frac{e^{j5\omega}}{\sqrt{2\pi}}\operatorname{sinc}(\omega) \quad \text{and} \quad \frac{e^{-j3\omega}}{\sqrt{2\pi}}\operatorname{sinc}(\omega)}$$

Frequency shifting

If $\mathscr{F}(f(t)) = F(\omega)$ then $\mathscr{F}\left(f(t)e^{j\omega_0 t}\right) = F(\omega-\omega_0)$

Example

If the Fourier transform of $f(t)$ is $F(\omega)$ then the transform of $f(t)\cos 4t$ is

....

32

$$\boxed{\frac{1}{2}(F(\omega+4)+F(\omega-4))}$$

Because

$$\begin{aligned} f(t)\cos 4t &= f(t)\frac{e^{j4t}+e^{-j4t}}{2} \\ &= \frac{1}{2}f(t)e^{j4t}+\frac{1}{2}f(t)e^{-j4t} \\ &= \frac{1}{2}\left(f(t)e^{j4t}+f(t)e^{-j4t}\right) \end{aligned}$$

and so the Fourier transform is $\frac{1}{2}(F(\omega-4)+F(\omega+4))$ by the linearity and the frequency shifting properties.

Time scaling

If $\mathscr{F}(f(t))=F(\omega)$ then

$$\mathscr{F}(f(kt))=\frac{1}{|k|}F\left(\frac{\omega}{k}\right)$$

So, for example, given $f(t)=\Pi_a(t)$ with Fourier transform $F(\omega)$, if $f(t)$ is shrunk to half its width then $F(\omega)$ is stretched to twice its width but shrunk to half its height.

Example

If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $f(-t)$ is ....

33

$$\boxed{F(-\omega)}$$

Because

$|k|^{-1}F(\omega/k)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(kt)e^{-j\omega t}\,dt$ and when $k=-1$ then

$$|-1|^{-1}F(\omega/[-1])=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-t)e^{-j\omega t}\,dt=F(-\omega)$$

Symmetry

If $\mathscr{F}(f(t))=F(\omega)$ then $\mathscr{F}(F(t))=f(-\omega)$

Example

The Fourier transform of $f(t)=\Pi_2(t)$ is $F(\omega)=\dfrac{1}{\sqrt{2\pi}}\text{sinc}(\omega)$, so the Fourier transform of

$$F(t)=\frac{1}{\sqrt{2\pi}}\text{sinc}(t) \text{ is } \ldots\ldots\ldots\ldots$$

34

$$\boxed{f(-\omega) = -\Pi_2(\omega)}$$

Because

The Fourier transform of $F(t) = \dfrac{1}{\sqrt{2\pi}}\text{sinc}\,(t)$

is $f(-\omega) = \Pi_2(-\omega) = -\Pi_2(\omega)$

Try one yourself.

Example

The Fourier transform of the unit constant function $f(t) = 1$ is

$$\mathcal{F}[1] = \ldots\ldots\ldots\ldots$$

35

$$\boxed{\sqrt{2\pi}\delta(\omega)}$$

Because

$$\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}} \text{ so } \mathcal{F}\left[\frac{1}{\sqrt{2\pi}}\right] = \delta(\omega), \text{ therefore } \mathcal{F}[1] = \sqrt{2\pi}\delta(\omega)$$

Differentiation

If $f(t) \to 0$ as $t \to \pm\infty$ and if $\mathcal{F}(f(t)) = F(\omega)$ then

$$\mathcal{F}(f'(t)) = \ldots\ldots\ldots\ldots$$

36

$$\boxed{j\omega F(\omega)}$$

Because

$$\begin{aligned}\mathcal{F}[f'(t)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)e^{-j\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[f(t)e^{-j\omega t}\right]_{-\infty}^{\infty} + \frac{j\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ &= 0 + j\omega F(\omega)\end{aligned}$$

In general, if $f(t) \to 0$ as $t \to \pm\infty$ and if $\mathcal{F}(f(t)) = F(\omega)$ then

If $\mathcal{F}(f(t)) = F(\omega)$ then $\mathcal{F}\left(f^{(n)}(t)\right) = (j\omega)^n F(\omega)$

where the superscript (n) indicates the nth derivative.

Example

The differential equation for unforced and undamped harmonic motion is of the form $mf''(t) + kf(t) = 0$. If we take the Fourier transform of this equation we immediately find that the permitted frequencies of oscillation are

$$\omega = \ldots\ldots\ldots\ldots$$

37

$$\boxed{\omega = \pm\sqrt{\frac{k}{m}}}$$

Because

If $F(\omega)$ is the Fourier transform of $f(t)$ then taking the Fourier transform of both sides of the equation $mf''(t) + kf(t) = 0$ gives by the differentiation property

$$m(j\omega)^2F(\omega) + kF(\omega) = (-m\omega^2 + k)F(\omega) = 0$$

so if $F(\omega) \neq 0$ then $m\omega^2 = k$ and so the permitted frequencies are

$$\omega = \pm\sqrt{\frac{k}{m}}.$$

Next frame

The Heaviside unit step function

38

The Heaviside unit step function is defined as $u(t)$ where

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

If we follow the definition of the Fourier transform we find that

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t)e^{-j\omega t}\,dt$$

So that $F(\omega) = \ldots\ldots\ldots\ldots$

39

$$\boxed{F(\omega) = \frac{1}{\sqrt{2\pi}j\omega} - \left\{1 - \underset{t\to\infty}{Lim}\left[e^{-j\omega t}\right]\right\}}$$

Because

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t)e^{-j\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-j\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}j\omega} - \left\{1 - \underset{t\to\infty}{Lim}\left[e^{-j\omega t}\right]\right\} \end{aligned}$$

Because $e^{-j\omega t} = \cos\omega t - j\sin\omega t$ we cannot say what happens to the exponential at $t \to \infty$. So how do we resolve the problem?

Next frame

40

Let $\mathscr{F}u(t) = F(\omega)$ and so, by the scaling property,$\mathscr{F}u(-t) = F(-\omega)$. Now, $u(t) + u(-t) = 1$, therefore $\mathscr{F}[u(t)] + \mathscr{F}u[(-t)] = \mathscr{F}[1]$. That is, from Frame 35

$$F(\omega) + F(-\omega) = \sqrt{2\pi}\delta(\omega)$$

We now assume that $F(\omega)$ consists of a combination of the Dirac delta and an arbitrary function $G(\omega)$

$$F(\omega) = \alpha\delta(\omega) + G(\omega) \text{ so that}$$

$$\begin{aligned} F(\omega) + F(-\omega) &= \alpha\delta(\omega) + G(\omega) + \alpha\delta(-\omega) + G(-\omega) \\ &= 2\alpha\delta(\omega) + G(\omega) + G(-\omega) \qquad \text{since } \delta(-\omega) = \delta(\omega) \\ &= \sqrt{2\pi}\delta(\omega) \end{aligned}$$

Therefore $\alpha = \sqrt{\dfrac{\pi}{2}}$ and $G(\omega) + G(-\omega) = 0$. That is, $G(\omega) = -G(-\omega)$.

Consequently $\mathscr{F}[u(t)] = F(\omega) = \sqrt{\dfrac{\pi}{2}}\delta(\omega) + G(\omega)$.

Now, $\mathscr{F}[u'(t)] = j\omega F(\omega) = j\omega\left\{\sqrt{\dfrac{\pi}{2}}\delta(\omega) + G(\omega)\right\}$ and since $u'(t) = \delta(t)$

then $\mathscr{F}[u'(t)] = \mathscr{F}[\delta(t)] = \dfrac{1}{\sqrt{2\pi}}$ giving $j\omega\left\{\sqrt{\dfrac{\pi}{2}}\delta(\omega) + G(\omega)\right\} = \dfrac{1}{\sqrt{2\pi}}$

Since $\omega\delta(\omega) = 0$, then $j\omega G(\omega) = \dfrac{1}{\sqrt{2\pi}}$ and so $G(\omega) = \dfrac{1}{j\omega\sqrt{2\pi}}$ thereby giving

$$\mathscr{F}[u(t)] = \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\}$$

The next property deals with the Fourier transform of a **product of functions** but before we go any further we need to discuss what is meant by the **convolution of two functions**.

Next frame

Convolution

41

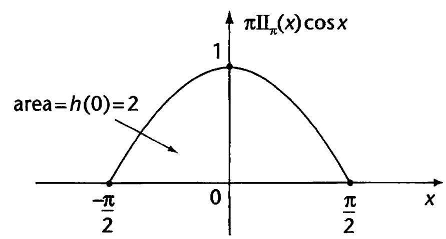
The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x = h(t)$$

where the $*$ denotes the operation of convolution. You will note that this is a function of variable t and here we have denoted it by $h(t)$. To illustrate an interpretation of this operation, consider the two functions $f(t) = \pi\Pi_\pi(t)$ and $g(t) = \cos t$. Then

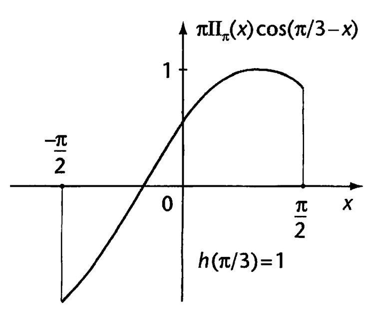
$$f(x)g(-x) = \pi\Pi_\pi(x)\cos(-x) = \pi\Pi_\pi(x)\cos x$$

is that part of the cosine function that lies between $\pm\pi/2$ and is zero elsewhere.



The integral $\int_{-\infty}^{\infty} f(x)g(-x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \pi\Pi_\pi(x)\cos x\,\mathrm{d}x = 2$. This is the area beneath the single loop of the cosine curve. We shall call this value $h(0)$ because the loop is centred on the origin. That is $h(0) = 2$. Now, the graph of $\cos(\pi/3 - x)$ has the same shape as $\cos(-x) = \cos x$ but it is shifted to the right by $\pi/3$ radians. Consequently, $f(x)g(\pi/3 - x) = \pi\Pi_\pi(x)\cos(\pi/3 - x)$ is that part of the *shifted* cosine function that lies between $\pm\pi/2$ and zero elsewhere, so now $\int_{-\infty}^{\infty} f(x)g(\pi/3 - x)\,\mathrm{d}x = 1$. We shall call this value $h(\pi/3)$ because $\pi/3$ measures the amount of the shift of the cosine curve. That is $h(\pi/3) = 1$. Proceeding in this manner to define values of $h(t)$ we see that the function formed from these integrals is

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x = h(t)$$



▶

Example

To find the convolution $f(t) * g(t)$ where

$$f(t) = u(t) \text{ and } g(t) = \begin{cases} \sec^2 t & |t| < \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

where $u(t)$ is the Heaviside function

then

$$h(t) = f(t) * g(t) = \dots\dots\dots\dots$$

42

$$\boxed{\frac{1 + \tan^2 t}{1 + \tan t}}$$

Because

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)g(t-x)\,dx &= \int_{-\infty}^{\infty} u(x)g(t-x)\,dx \\ &= \int_0^{\pi/4} \sec^2(t-x)\,dx \quad \text{because } u(t) = 0 \text{ for } t < 0 \text{ and } g(t) = 0 \text{ for } t > \pi/4 \\ &= \Big[-\tan(t-x)\Big]_0^{\pi/4} \\ &= \{-\tan(t - \pi/4) + \tan t\} \\ &= -\frac{\tan t - 1}{1 + \tan t} + \tan t \\ &= \frac{1 + \tan^2 t}{1 + \tan t} \end{aligned}$$

Next frame

The convolution theorem

43

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively then

(a) The Fourier transform of the convolution of $f(t)$ and $g(t)$ is equal to the product of the individual Fourier transforms. That is

$$\mathcal{F}[f(t) * g(t)] = \sqrt{2\pi}F(\omega)G(\omega) \text{ and so}$$

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = \frac{1}{\sqrt{2\pi}}[f(t) * g(t)]$$

(b) The Fourier transform of the product $f(t)g(t)$ is equal to the convolution of the individual Fourier transforms. That is

$$\mathcal{F}[f(t)g(t)] = \frac{1}{\sqrt{2\pi}}F(\omega) * G(\omega) \text{ and so}$$

$$\mathcal{F}^{-1}[F(\omega) * G(\omega)] = \sqrt{2\pi}f(t)g(t)$$

These provide useful methods of finding inverse transforms.

Example

To find the inverse transform of

$$F(\omega) = \frac{1}{2\pi(a+j\omega)^2} = \frac{1}{\sqrt{2\pi}(a+j\omega)} \times \frac{1}{\sqrt{2\pi}(a+j\omega)} \text{ where } a > 0$$

we note that if $F_1(\omega) = \dfrac{1}{\sqrt{2\pi}(a+j\omega)}$ then from Frame 26

$$f_1(t) = \mathscr{F}^{-1}[F_1(\omega)] = \dots\dots\dots\dots$$

44

$$\boxed{f_1(t) = e^{-at}u(t)}$$

Now, because

$$F(\omega) = F_1(\omega)F_1(\omega)$$

then

$$\begin{aligned} f(t) = \mathscr{F}^{-1}[F(\omega)] &= \mathscr{F}^{-1}[F_1(\omega)F_1(\omega)] = \frac{1}{\sqrt{2\pi}}[f_1(t) * f_1(t)] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)f_1(t-x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax}u(x)e^{-a(t-x)}u(t-x)\,\mathrm{d}x \\ &= \frac{e^{-at}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax}u(x)e^{ax}u(t-x)\,\mathrm{d}x \\ &= \frac{e^{-at}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)u(t-x)\,\mathrm{d}x \end{aligned}$$

Now, $u(x)u(t-x) = 0$ when $x < 0$ or when $t - x < 0$, that is when $x > t$.

Therefore, $u(x)u(t-x) = \begin{cases} 1 & \text{if } 0 < x < t \\ 0 & \text{otherwise} \end{cases}$ so

$$\begin{aligned} f(t) &= \frac{e^{-at}}{\sqrt{2\pi}}\int_0^t \mathrm{d}x \\ &= \begin{cases} \dfrac{te^{-at}}{\sqrt{2\pi}} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \text{that is, } f(t) = \frac{te^{-at}}{\sqrt{2\pi}}u(t) \end{aligned}$$

Now you try one.

The inverse Fourier transform of $F(\omega) = \dfrac{5}{6 + 5j\omega - \omega^2}$ is

$$f(t) = \dots\dots\dots\dots$$

45

$$\boxed{f(t) = \sqrt{50\pi}[e^{-2t} - e^{-3t}]u(t)}$$

Because

$$F(\omega) = \frac{5}{6 + 5j\omega - \omega^2}$$

$$= \frac{5}{(2 + j\omega)(3 + j\omega)}$$

Let $F_1(\omega) = \dfrac{1}{\sqrt{2\pi}(2 + j\omega)}$ so that $f_1(t) = e^{-2t}u(t)$ and

$F_2(\omega) = \dfrac{1}{\sqrt{2\pi}(3 + j\omega)}$ so that $f_2(t) = e^{-3t}u(t)$ so that

$$F(\omega) = 10\pi[F_1(\omega)F_2(\omega)]$$

By the convolution theorem

$$f(t) = \frac{10\pi}{\sqrt{2\pi}}[f_1(t) * f_2(t)]$$

$$= \sqrt{50\pi}\int_{-\infty}^{\infty} f_1(x)f_2(t - x)\,dx$$

$$= \sqrt{50\pi}\int_{-\infty}^{\infty} e^{-2x}u(x)e^{-3(t-x)}u(t - x)\,dx$$

$$= \sqrt{50\pi}e^{-3t}\int_{-\infty}^{\infty} e^{x}u(x)u(t - x)\,dx$$

$$= \sqrt{50\pi}e^{-3t}\int_{0}^{t} e^{x}\,dx \text{ since } u(x)u(t - x) = \begin{cases} 1 & \text{if } 0 < x < t \\ 0 & \text{otherwise} \end{cases}$$

$$= \sqrt{50\pi}e^{-3t}\left[e^{t} - 1\right]u(t) \text{ since } \int_{0}^{t} e^{x}\,dx = \begin{cases} e^{t} - 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$= \sqrt{50\pi}\left[e^{-2t} - e^{-3t}\right]u(t)$$

Move to the next frame

Fourier cosine and sine transforms

46

Given that

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt$$

if $f(t)$ is an even function so that $f(-t) = f(t)$ then

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt \quad \text{since } f(t)\sin\omega t \text{ is odd}$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos\omega t\,dt$$

This is referred to as the Fourier cosine transformation and is denoted by $F_c(\omega)$. That is

$$F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos\omega t\,dt$$

Similarly if $f(t)$ is an odd function so that $f(-t) = -f(t)$ then

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt$$

$$= \frac{j}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt \quad \text{since } f(t)\cos\omega t \text{ is odd}$$

$$= j\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t\,dt$$

This gives rise to the Fourier sine transformation, denoted by $F_s(\omega)$ where

$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t\,dt$$

Example 1

The Fourier cosine transformation of $f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ 0 & \text{if } t \geq a \end{cases}$ is

$F_c(\omega) = \ldots\ldots\ldots\ldots$

47

$$\boxed{F_c(\omega) = \sqrt{\frac{2}{\pi}}a \,\mathrm{sinc}\,(\omega a)}$$

Because

$$\begin{aligned}F_c(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\omega t\,dt\\ &= \sqrt{\frac{2}{\pi}}\int_0^{a} \cos\omega t\,dt\\ &= \sqrt{\frac{2}{\pi}}\left[\frac{\sin\omega t}{\omega}\right]_0^a\\ &= \sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{\omega}\right) = \sqrt{\frac{2}{\pi}}a\mathrm{sinc}\,(\omega a)\end{aligned}$$

Example 2

The Fourier sine transformation of $f(t) = \begin{cases}1 & \text{if } 0 < t < a\\ 0 & \text{if } t \geq a\end{cases}$ is

$F_s(\omega) = \ldots\ldots\ldots\ldots$

48

$$\boxed{F_s(\omega) = \sqrt{\frac{2}{\pi}}2a^2\omega\,\mathrm{sinc}^2(\omega a)}$$

Because

$$\begin{aligned}F_s(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\,dt\\ &= \sqrt{\frac{2}{\pi}}\int_0^{a} \sin\omega t\,dt\\ &= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos\omega t}{\omega}\right]_0^a\\ &= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos\omega a}{\omega}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{2\sin^2\omega a}{\omega}\right) = \sqrt{\frac{2}{\pi}}2a^2\omega\,\mathrm{sinc}^2(\omega a)\end{aligned}$$

The Fourier cosine and sine transforms are useful when $f(t)$ is only defined for $t \geq 0$ and where an extension can be added to $f(t)$ for $t < 0$ that makes the extended $f(t)$ into an even or odd function respectively.

Table of transforms

49

$f(t) = \begin{cases} 1 & \text{if } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$	$F(\omega) = \dfrac{a}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)$
$f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ 0 & \text{otherwise} \end{cases}$	$F(\omega) = \dfrac{a e^{-j\omega a/2}}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)$
$\Pi_a(t) = \begin{cases} 1/a & \text{if } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)$
$f(t) = u(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \left\{ \pi\delta(\omega) + \dfrac{1}{j\omega} \right\}$
$f(t) = e^{-at}u(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \left\{ \pi\delta(\omega + a) + \dfrac{1}{j\omega} \right\}$
$f(t) = te^{-at}u(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}(a + j\omega)^2}$
$f(t) = \delta(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}}$

The main points of the Programme are listed in the **Revision summary** that follows. Read it in conjunction with the **Can You?** checklist and refer back to the relevant parts of the Programme, if necessary. You will then have no trouble with the **Test exercise** and the **Further problems** provide valuable additional practice.

Revision summary 7

50

1 *Complex Fourier series*

The Fourier series of the piecewise continuous function $f(t)$ with piecewise continuous derivative and where $f(t+T) = f(t)$ is given as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $c_n = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$.

2 *Discrete complex spectra*
The c_n are complex numbers and can be written as

$$c_n = |c_n|e^{j\phi_n}$$

These complex coefficients constitute a discrete complex spectrum where c_n represents the *spectral coefficient* of the nth harmonic. Each spectral coefficient couples an amplitude spectrum value $|c_n|$ and a phase spectrum value ϕ_n.

3 *Fourier's integral theorem*
If (a) $f(t)$ and $f'(t)$ are piecewise continuous in every finite interval

(b) $f(t)$ is absolutely integrable in $(\infty, -\infty)$, that is $\int_{-\infty}^{\infty} |f(t)|\,dt$ is finite then

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt.$$

4 *Continuous complex spectra*
The Fourier transform $F(\omega)$ is a complex function so we write $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ where $|F(\omega)| = \left|\left(a/\sqrt{2\pi}\right)\operatorname{sinc}(\omega a/2)\right|$ is the *continuous amplitude spectrum* and $\phi(\omega) = -a\omega/2$ is the *continuous phase spectrum.*

5 *Transforms of special functions*

Top-hat function

$$\Pi_a(t) = \begin{cases} 1/a & -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$$

with Fourier transform $F(\omega) = \dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}(\omega a/2)$.

The Dirac delta

If $f(t) = \delta(t)$ then $F(\omega) = \dfrac{1}{\sqrt{2\pi}}$.

The Heaviside unit step function

$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$ has the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega + a) + \frac{1}{j\omega}\right\}.$$

The triangle function

$\Lambda(t) = \begin{cases} 0 & |t| > 1 \\ 1 & |t| < 1 \end{cases}$ has the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega/2).$$

▶

6 *Alternative forms*

There are a number of alternative forms for the Fourier transform – each dealing with a different location for the constant 2π. Other forms are

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \text{ or}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \text{ or}$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j2\pi\omega t}\,d\omega \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\omega t}\,dt.$$

7 *Properties of the Fourier transform*

Time shifting

$$\text{If } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt \text{ then}$$

$$e^{j\omega t_0}F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-t_0)e^{j\omega t}\,dt.$$

Linearity

If $F_1(\omega)$ and $F_2(\omega)$ are the Fourier transforms of $f_1(t)$ and $f_2(t)$ respectively then $\alpha_1F_1(\omega) + \alpha_2F_2(\omega)$ is the Fourier transform of $\alpha_1f_1(t) + \alpha_2f_2(t)$ where α_1 and α_2 are constants.

Frequency shifting

If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $f(t)e^{-j\omega_0 t}$ is $F(\omega - \omega_0)$.

Time scaling

$$\text{If } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt \text{ then}$$

$$|k|^{-1}F(\omega/k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(kt)e^{j\omega t}\,dt.$$

Symmetry

If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $F(t)$ is $f(-\omega)$.

Differentiation

$$\text{If } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \text{ then}$$

$$(j\omega)^nF(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{(n)}(t)e^{-j\omega t}\,dt \text{ and}$$

$$F^{(n)}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-j\omega)^nf(t)e^{-j\omega t}\,dt.$$

▶

8 *Convolution*

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x = h(t).$$

The convolution theorem

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively then

(a) The Fourier transform of the convolution of $f(t)$ and $g(t)$ is equal to the product of the individual Fourier transforms. That is

$$\mathscr{F}[f(t) * g(t)] = F(\omega)G(\omega).$$

(b) The Fourier transform of the product $f(t)g(t)$ is equal to the convolution of the individual Fourier transforms. That is

$$\mathscr{F}[f(t)g(t)] = F(\omega) * G(\omega).$$

9 *Fourier cosine and sine transforms*

Given that $f(t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}t$ where

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}\omega$$

where $f(t)$ is even then

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(\omega)\cos\omega t\,\mathrm{d}\omega \text{ where } F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\omega t\,\mathrm{d}t$$

and where $F_c(\omega)$ is called the *Fourier cosine transformation*. This transformation is useful when $f(t)$ is defined only for $t \geq 0$ and where an extension can be added to $f(t)$ for $t < 0$ that makes the extended $f(t)$ into an even function.

If $f(t)$ is odd then

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(\omega)\sin\omega\,\mathrm{d}\omega \text{ where } F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\,\mathrm{d}t$$

and where $F_s(\omega)$ is called the *Fourier sine transformation*. This transformation is useful when $f(t)$ is defined only for $t \geq 0$ and where an extension can be added to $f(t)$ for $t < 0$ that makes the extended $f(t)$ into an odd function.

Can You?

Checklist 7

51

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Convert a trigonometric Fourier series into a doubly infinite sum of complex exponentials? 1 to 4

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the complex Fourier series of a function that satisfies Dirichlet's conditions? 5 to 8

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the function sinc(t)? 9

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Separate a discrete complex spectrum into an amplitude spectrum and a phase spectrum? 10 to 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- State Fourier's integral theorem in terms of complex exponentials? 13 to 15

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Define and derive the Fourier transform of a function satisfying Dirichlet's conditions? 16 and 17

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Separate a continuous complex spectrum into an amplitude spectrum and a phase spectrum? 18

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the functions $\Pi_a(t)$ and $\Lambda_a(t)$ and derive their Fourier transforms along with those of the Dirac delta and the Heaviside unit step? 19 to 27

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise alternative forms of the function–transform pair? 28

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Reproduce a collection of properties of the Fourier transform? 29 to 40

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate the convolution of two functions and describe its Fourier transform? 41 to 45

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Fourier sine and cosine transformations? 46 to 48

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 7

52

1 Find the complex Fourier series of the sawtooth wave
$f(t) = t, \quad 0 < t < 1 \quad$ and where $\quad f(t+1) = f(t)$.

2 Find the Fourier transform of

$$f(t) = \begin{cases} e^{-at} & |t| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad a > 0$$

3 Given that the Dirac delta $\delta(t)$ has the Fourier transform $F(\omega) = \dfrac{1}{\sqrt{2\pi}}$, show, by considering the inverse Fourier transform, that

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega t\,d\omega.$$

4 If $f(t)$ and $F(\omega)$ form a Fourier transform pair, find the Fourier transform of $f(t)\sin\omega_0 t$ where ω_0 is a constant.

5 Find the inverse transform of $F(\omega) = \dfrac{6}{\omega^2 + 5j\omega - 4}$.

6 Find the Fourier sine and cosine transformations of $f(t) = e^{-kt}$ for $t > 0$ and $k > 0$.

Further problems 7

53

1 By comparing the trigonometric Fourier series of a periodic function with its complex exponential counterpart show that

$$|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \text{ and } \phi_n = \arctan\left\{-\frac{b_n}{a_n}\right\} \text{ where } c_n = |c_n|e^{j\phi_n}.$$

2 Prove Parseval's identity for the periodic function with period T

$$\frac{1}{T}\int_{-T/2}^{T/2} \{f(t)\}^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=-\infty}^{\infty}(a_n^2 + b_n^2)$$

and show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

3 Draw the graph and find the complex Fourier series of the rectified sine wave

$f(t) = \sin\pi t, \quad 0 < t < 1 \quad$ where $f(t+1) = f(t)$.

4 Draw the graph and find the complex Fourier series of the rectified cosine wave

$f(t) = \cos\pi t, \quad -1/2 < t < 1/2 \quad$ where $f(t+1) = f(t)$.

5 Draw the graph and find the complex Fourier series of

$$f(t) = e^{\pi t}, \quad 0 < t < 2 \quad \text{where } f(t+2) = f(t).$$

6 Draw the graph and find the complex Fourier series of the sawtooth wave

$$f(t) = -\frac{t}{T} + \frac{1}{2}, \quad 0 < t < T \quad \text{where } f(t+T) = f(t).$$

7 If $f_1(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$ and $f_2(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega_0 t}$ where $\omega_0 = 2\pi/T$, show that the convolution

$$f_1(t) * f_2(t) = \sum_{n=-\infty}^{\infty} c_n d_n e^{jn\omega_0 t}.$$

8 Find the Fourier transform of

$$f(t) = \begin{cases} \cosh t & \text{for } |t| < 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

9 Find the Fourier transform of

$$f(t) = \begin{cases} \sinh t & \text{for } |t| < 1 \\ 0 & \text{for } |t| > 1. \end{cases}$$

10 Find the Fourier transform of

$$f(t) = \begin{cases} \sin \pi t & \text{for } 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

11 Find the Fourier transform of

$$f(t) = \begin{cases} \cos \pi t & \text{for } |t| < 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

12 Draw the graph and find the Fourier transform of

$$f(t) = e^{-a|t|}, \quad a > 0.$$

13 Given that

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 0 \\ -1 & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Draw the graph of $f(t)$

(b) Express $f(t)$ in terms of the Heaviside unit step function

(c) Find the Fourier transform of $f(t)$.

14 Draw the graph and find the Fourier transform of

$$f(t) = (u(t) - u(t - \pi)) \cos kt.$$

15 Show that if $f(t)$ is real then the corresponding Fourier transform $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ is such that $|F(\omega)|$ is even and $\phi(\omega)$ is odd.

▶

16 Show that if the Fourier transform of a real function is real then $f(t)$ is even, and if the Fourier transform of a real function is imaginary then $f(t)$ is odd.

17 Defining the squared modulus of the Fourier transform $|F(\omega)|^2 = F(\omega)F^*(\omega)$ where $F^*(\omega)$ is the complex conjugate of $F(\omega)$, prove Parseval's theorem

$$\int_{-\infty}^{\infty} [f(t)]^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} |F(\omega)|^2 \, \mathrm{d}t.$$

18 Show that the convolution of a top-hat function with itself is the triangle function. That is

$$\Pi_a(t) * \Pi_a(t) = \Lambda_a(t).$$

19 Show that $\operatorname{sinc}(t) * \operatorname{sinc}(t) = \operatorname{sinc}(t)$.

20 Find the Fourier sine and cosine transforms of

$$f(t) = \begin{cases} e^{at} & \text{for } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

21 Find the Fourier sine and cosine transforms of

$$f(t) = \begin{cases} \cosh t & \text{for } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$